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Abstract

We introduce a general random pseudofractal network model by assigning
fitness to each edge. In this model, continuous growth and attachment,
determined by their fitness of already existing edges, are the two ingredients.
We obtain the analytical results that our model exhibits a power-law degree
distribution with exponent $\gamma = 2 + m(1 + \alpha m)^{-1}$, where m and α are tunable
parameters. We also show that a general random pseudofractal network has
a large clustering coefficient and a small average distance leading to a small-
world effect. These theoretical results agree well with numerical simulations.

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1. Introduction

The past decade has witnessed a recognition that real-world networks arising in physical,
social and biological systems are fundamentally different from the random graph models
[1, 2]. Despite diversity, these networks generally exhibit statistical and topological properties.
The most relevant among these properties are small-world effect [3] and scale-free feature
[4]. The small-world effect implies that a network has a high degree of clustering and a
short average distance. And scale-free networks are characterized by a power-law degree
distribution $P(k) \sim k^{-\gamma}$, where k is the degree of a node and the exponent γ is a positive real
number, controlling the broadness of the distribution.

Motivated by these results, a considerable number of network models have been proposed
in order to capture the above properties [5–13]. Most are based on two ingredients originally
introduced by Barabási and Albert [4]: growth and preferential attachment. In Barabási–
Albert (BA) model, the network starts from a small number of nodes (m_0). At every time
step, a new node is added and connected to m ($m \leqslant m_0$) different nodes with the probability
linearly proportional to the degree of the target node. Such a growth rule generates a power-
law degree distribution with the exponent $\gamma = 3$. Actually, scale-free and small-world are
not exclusive in real-world networks. Much valuable work has been focused on this topic
[14–17]. In this endeavor, an interesting model, which is also an improvement of the BA model,

is called pseudofractal network (PN) based on the fact that the structure of such a network has no fixed finite fractal dimension. PN was first introduced by Dorogovtsev and Mendes *et al* owing to citation graphs [18, 19]. The growth starts from a single edge connecting two nodes. At each time step, a new node is added to every edge of the graph, which is attached to both end nodes of the edge. Then a deterministic network model with a fixed average degree is constructed. Subsequently, Comellas [20] and Zhang [21] *et al* extended the PN to a process of edge multiplication by a recursive method, respectively. Analytical results show that these deterministic models have small-world and scale-free properties simultaneously. Recent studies also show random PNs [22–24] have a large clustering coefficient, a small average distance and a power-law degree distribution. In particular, random PN has the same power-law exponent as the BA's, and the fact that a new node is connected to both ends of a randomly chosen edge by two new edges, implying the preferential attachment. However, the exponent of power-law distribution usually ranges between 2 and 3 in most real systems. The above models have difficulty in capturing the property. Wang *et al* [25] introduced fitness competition to random PNs, and deduced the degree distribution such that $P(k) \sim k^{-\gamma}$ for large k, where $\gamma = 2 + (1 + \alpha)^{-1}$, and α is the competition parameter. By adjusting α, one can obtain a variety of scale-free networks.

In this paper, we present a general random PN by a simple recursive rule. The proposed model is still called the pseudofractal network, stemming from the pioneering work of Dorogovtsev and Mendes. Also similar to the fractal, the mechanism of the general random PNs is recursive and self-similar. Of course, in general PNs, only one new type of structure appears, which is different from the fractal in all scales. The rest of this paper is organized as follows. Section 2 provides a detailed description of the general random PN model. In section 3, we give analytical and simulation results of the network properties including degree distribution, clustering coefficient and average distance. Finally, we draw the main conclusions in section 4.

2. A general random pseudofractal network model

In this section, we describe a model of growing network, which is constructed by a simple rule. We denote nodes in our network by N_i with $i = -1, 0, 1, 2, \ldots$, the edge between nodes N_p and N_q by $\overline{N_p N_q}$. In this network, all nodes are considered to be distributed on a ring. We denote by $\widehat{N_p N_q}$ an arc of the corresponding edge $\overline{N_p N_q}$, and l_{pq} the length of arc $\widehat{N_p N_q}$. Then $f_{pq} = l_{pq}^\alpha$ is defined as the fitness of an edge $\overline{N_p N_q}$ with $\alpha \geqslant 0$. As a matter of fact, all nodes will not be distributed on a ring since all edges in fractal are of the same length. Assigning all nodes onto the same ring is easy to understand the fitness and node adding evolution.

The network starts from an initial state of two nodes N_{-1}, N_0, connecting by an edge $\overline{N_{-1} N_0}$ at $t = 0$. At $t = 1$, m nodes orderly connected by $m + 1$ edges are attached to node N_0 and N_{-1}, forming an $(m + 2)$-regular polygon. Obviously, all nodes are distributed on a ring. For $t \geqslant 2$, an edge $\overline{N_i N_j}$ is selected with the probability linearly proportional to the edge's fitness, and m nodes orderly connected by $m + 1$ edges are added to both ends of $\overline{N_i N_j}$, where $\overline{N_i N_j}$ can be any edge present in the network at time t, the newly added edges whose nodes are all distributed on the same ring are of the same length. In particular, we label these newly added nodes as N_{i*T+t} for all $t \geqslant 1$, $i = 0, 1, 2, \ldots, m - 1$, and the arc length of the new edges as $l(t)$ at t time step, where T is a sufficiently large natural number. The growing process is repeated until the network reaches the desired size. Figure 1 shows a possible process of the general random PN with $m = 2$. Note that figure 1 is just a demonstration of fitness and network evolution, not the real case of the edge length.
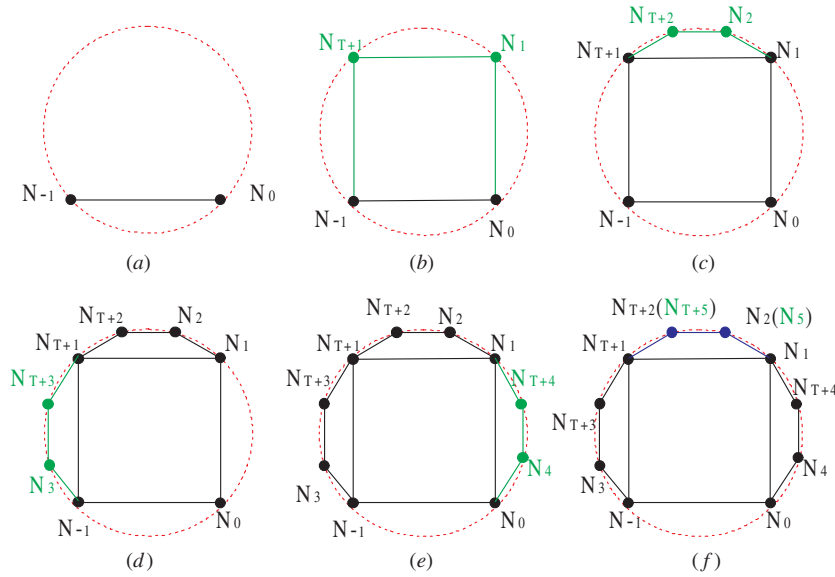
Figure 1. Scheme of a general random PN growth with $m = 2$. The figures from (a) to (f) show a possible process for the general random PN. The growth starts from an edge $\overline{N_{-1}N_0}$ with two nodes N_{-1}, N_0 at $t = 0$ (a). At the first time step $t = 1$, the edge $\overline{N_{-1}N_0}$ is selected. Nodes N_1, N_{T+1} and three edges $\overline{N_{-1}N_{T+1}}$, $\overline{N_0N_1}$, $\overline{N_1N_{T+1}}$ are all added to the growing network (b). Then we obtain a square and its circumcircle (red dashed line). At the second time step, the edge $\overline{N_{T+1}N_1}$ is selected with a certain probability, then nodes N_2, N_{T+2} and edges $\overline{N_{T+1}N_2}$, $\overline{N_{T+2}N_2}$, and $\overline{N_2N_1}$ are added to the third point of the arc $\widehat{N_1N_{T+1}}$ (c). Other sub-figures can be given in a similar way. With this rule, one can easily get a general random PN. Obviously, a third point may be selected more than once shown in (f).

(This figure is in colour only in the electronic version)

In a general random PN, the number of nodes and edges is $mt + 2$ and $(m + 1)t + 1$, respectively. Then the average degree of the whole network is equal to $2(m + 1)m^{-1}$ when t is big enough. Besides, all newly added arc lengths are the same and their sum is equal to the arc length of the selected edge at each time step. Therefore,

$$S_l(t + 1) = S_l(t) + l(v), \tag{1}$$

and

$$S_f(t + 1) = S_f(t) + (m + 1)\left(\frac{l(v)}{m + 1}\right)^\alpha, \tag{2}$$

where $l(v)$ is the length of the selected arc, $S_l(t)$ and $S_f(t)$ are the arc length sum and fitness sum at t time step, respectively. The general random PN will reduce to a special random PN with competition if $m = 1$. In this situation, fitness does no longer work, and all edges are of the same fitness. That is to say, fitness selection is replaced by preferential attachment, which is similar to the BA model.

3. Properties of the general PNs

The degree distribution, clustering coefficient and average distance are three important properties to understand the complex dynamics of real-life systems. In this section, we focus on the three parameters.

3.1. Degree distribution

Let the average arc length $\bar{l}(i)$ be the expectation value of $l(i)$, and the initial arc length be $l(0) = l(j * T + 1) = 1$ for all $j = 0, 1, 2, \ldots, m - 1$, where $l(0) = l_{-10}$. Then the average fitness at the ith time step satisfies

$$\bar{f}(i) = \bar{l}^\alpha(i). \tag{3}$$

According to the mechanism of the growing PN, edge $\overline{N_x N_y}$ already present in our network is selected in the probability $p_u(i)$ such that

$$p_u(i) = \frac{\bar{f}(u)}{\bar{S}_f(i)} \tag{4}$$

at time step i, where $u = \max\{x, y\}$, $\bar{S}_f(i) = 1 + (m + 1) \sum_{j=1}^{i-1} \bar{l}^\alpha(j)$ is the average fitness sum at time step i. Then the evolution equation of the average arc length can be written as

$$\bar{l}(t) = \sum_{j=1}^{t-1} \bar{l}(j) p_j(t). \tag{5}$$

Putting equations (4) and (5) together, we obtain

$$\bar{l}(t) = \frac{1 + (m + 1) \sum_{j=1}^{t-1} \bar{l}^{1+\alpha}(j)}{(m + 1)\left(1 + (m + 1) \sum_{j=1}^{t-1} \bar{l}^\alpha(j)\right)} \tag{6}$$

with $t = 2, 3, \ldots$ and $\bar{l}(1) = 0.5$. As has been shown in [25], equation (6) is rewritten in a continuous approximation as

$$\bar{l}(t) = \frac{\int_0^t \bar{l}^{1+\alpha}(v)\,\mathrm{d}v}{(m + 1) \int_0^t \bar{l}^\alpha(v)\,\mathrm{d}v}. \tag{7}$$

when time step t is sufficiently large and fitness sum $\bar{S}_f(t)$ satisfies the condition $\bar{S}_f(t) \gg 1$.

The following supplies the detailed solving process of the continuous evolution equation (7). It follows from equation (7) that the derivative of average arc length, $\dot{\bar{l}}(t)$, is satisfied with

$$\dot{\bar{l}}(t) = -\frac{m\bar{l}^{1+\alpha}(t)}{(m + 1) \int_0^t \bar{l}^\alpha(v)\,\mathrm{d}v}. \tag{8}$$

Substituting the integral term of equation (8) into equation (7), we have

$$\frac{\dot{\bar{l}}(t)}{\bar{l}(t)} = -\frac{m\bar{l}^{1+\alpha}(t)}{\int_0^t \bar{l}^{1+\alpha}(v)\,\mathrm{d}v}. \tag{9}$$

Let $g(t) = \int_0^t \bar{l}^{1+\alpha}(v)\,\mathrm{d}v$, it is straightforward to get $\dot{g}(t) = \bar{l}^{1+\alpha}(t)$ and $\dot{\bar{l}}(t) = (1 + \alpha)^{-1} \bar{l}^{-\alpha}(t) \ddot{g}(t)$, where $\ddot{g}(t)$ is the derivative of $\dot{g}(t)$. Substituting these expressions into equation (9) yields

$$\frac{\ddot{g}(t)}{\dot{g}(t)} = -m(\alpha + 1)\frac{\dot{g}(t)}{g(t)}. \tag{10}$$

Applying $\int_0^t \mathrm{d}v$ to both sides of equation (10) twice, one gets $g(t) = [(m(\alpha + 1) + 1)t]^{\frac{1}{m(\alpha+1)+1}}$. Thus we have

$$\bar{l}(t) = \left(\frac{\beta}{mt}\right)^\beta, \tag{11}$$

and

$$\overline{S}_f(t) = \left(\frac{\beta}{mt}\right)^{\alpha\beta-1},$$ (12)

where initial values are selected as $g(0) = \dot{g}(0) = 1$, and $\beta = m(m(\alpha+1)+1)^{-1}$. Obviously, the fitness sum $\overline{S}_f(t) \gg 1$ holds since α and m are both positive constants. Also note that the exact expression of $\bar{l}(t)$ is of the following form,

$$\bar{l}(t) = c_1 \left(\frac{\beta}{m(t+c_2)}\right)^\beta,$$ (13)

where c_1 and c_2 are constants determined by the initial values. Actually, the constant c_1, no matter what value is selected, will not affect the following results. And the constant c_2 can be ignored when t is large enough. Therefore, the following derivation involves approximation in equation (11) for simplification.

Another important parameter used to compute degree distribution is average node strength. Denote the average strength of node N_i by $\bar{s}(i,t)$ at time t, and $\bar{s}(i,t)$ is defined as the fitness sum of edges that connects to node N_i. According to the evolution of general random PNs, we have

$$\frac{\partial \bar{s}(i,t)}{\partial t} = \frac{\bar{s}(i,t)\overline{f}(t)}{\overline{S}_f(t)},$$ (14)

accounting for the boundary condition $\bar{s}(i,i) = 2\overline{f}(i)$. Recalling equation (11), we easily derive the solution

$$s(i,t) = 2\beta^{\alpha\beta} m^{-\alpha\beta} i^{-\frac{\beta(m\alpha+1)}{m}} t^{\frac{\beta}{m}}.$$ (15)

Hereby, we can compute the degree distribution from continuous expressions of average arc length, fitness sum and node strength. The degree distribution of the node N_i in the continuous approximation is $p(k,i,t) = \delta(k - \bar{k}(i,t))$, where $\delta(\cdot)$ is the δ-function, and $\bar{k}(i,t)$ is the average degree of the node N_i at time t. Then we have

$$\frac{\partial \bar{k}(i,t)}{\partial t} = \frac{\bar{s}(i,t)}{\overline{S}_f(t)},$$ (16)

where $\bar{k}(i,i) = 2$. Applying $\int_i^t \mathrm{d}t$ to both sides of equation (16), we obtain the solution,

$$\bar{k}(i,t) = \frac{2}{1+\alpha\beta}\left(\frac{t}{i}\right)^{\frac{\beta(1+\alpha m)}{m}} - \frac{2}{1+\alpha\beta} + 2.$$ (17)

In the continuous approximation [10], one can find the distribution $P(k,t)$ using the derived $\bar{k}(i,t)$,

$$P(k,t) = -\frac{1}{t}\left(\frac{\partial \bar{k}(i,t)}{\partial i}\right)^{-1}.$$ (18)

Therefore, the degree distribution can be written as

$$P(k,t) = (\gamma-1)\left(\frac{1+\alpha\beta}{2}\right)^{1-\gamma}\left(k - \frac{2\alpha\beta}{1+\alpha\beta}\right)^{-\gamma},$$ (19)

where $\gamma = 2 + m(1+\alpha m)^{-1}$. Thus, in the region $k \gg 2\alpha\beta(1+\alpha\beta)^{-1}$, we obtain the stationary degree distribution, $P(k) \sim k^{-\gamma}$ confirming a power-law. If selecting $m = 1$, the power-law exponent is $\gamma = 2 + (\alpha+1)^{-1}$ in accordance with the results in [25]. Obviously, by tuning the parameters α and m, it is possible to obtain a variety of scale-free networks with different exponents in the range, $\gamma > 2$. In figure 2, we report the simulation results of the degree distribution for several values of α and m. From figure 2, we can see that the degree distribution follows a power-law for large degree values, which agrees well with the analytical results.
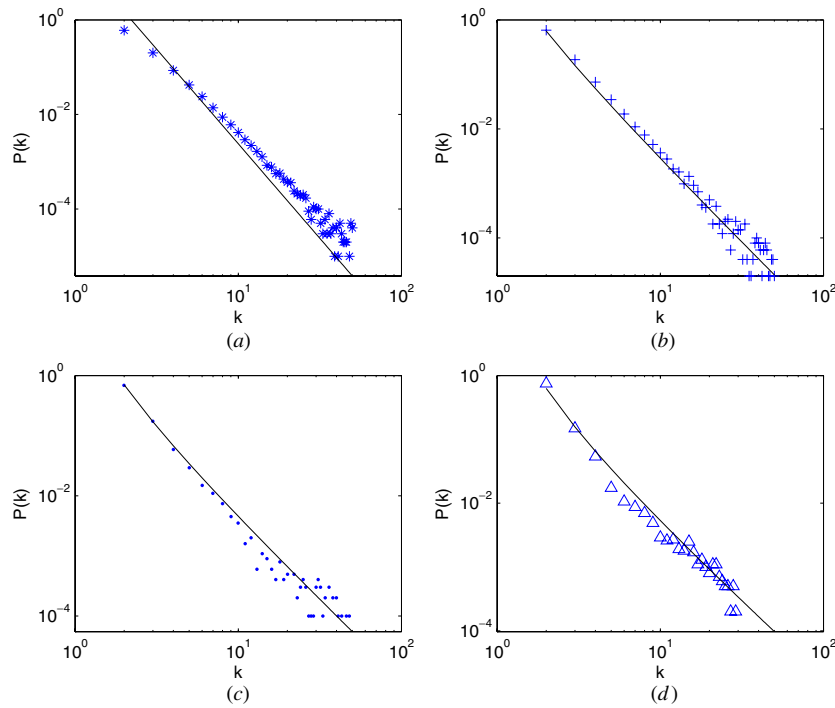
Figure 2. The degree distribution of the general random PNs, where the network size $N = 10^5, m = 2$. The four sub-figures show the results with $\alpha = 0$ (∗, a), $\alpha = 0$ (+, b), $\alpha = 0$ (·, c) and $\alpha = 0$ ($\triangle$, d). In these figures, the curves (black) are the theoretical results, while the marks (∗, +, · and $\triangle$) are the simulation results. Obviously, all degree distributions follow a power-law.

3.2. Clustering coefficient

A remarkable difference between the random PNs with competition and the proposed model in this paper is the size of the smallest loops. For example, the smallest loop in random PNs with competition is triangle, while the general model can be represented by a grid-like frame or a more complex structure. Obviously, the clustering coefficient, defined as the fraction of triangles present in the network, is unable to quantify the proposed network's structure. Then we define a *clustering coefficient of order x* for a node N_i as the probability that there is a distance of length x between two neighbors of a node N_i, namely, the higher order clustering coefficients follow [26, 27]:

$$C_i(x) = \frac{2E_i(x)}{k_i(k_i - 1)}, \tag{20}$$

where k_i denotes the degree of node N_i, and $E_i(x)$ is the number of x-distance between two neighbors of node N_i. The clustering coefficient $C(x)$ is the average value of $C_i(x)$ over the whole network, i.e.,

$$C(x) = \frac{1}{N} \sum_{i=1}^{N} C_i(x), \tag{21}$$

where N is the number of nodes. Note that $\sum_x C(x) = 1$ and $C(x)$ will reduce to the standard clustering coefficient for $x = 1$.
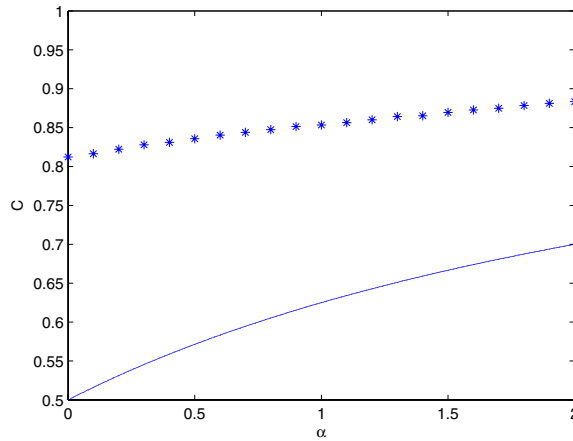
Figure 3. The clustering coefficients of the general random PNs for various α, where the network size $N = 10^5$, $m = 2$. The stars and the curve are the simulation and analytical results, respectively. From the figure, we can see that the two results almost have the similar tendency as α increases to 2. Yet there exist quite large errors between the numerical simulations and analytical results derived from the degree distribution in equation (19).

In general random PNs, E_i will increase by 1 if the degree of a node N_i increases by 1 at a certain time step. Then we can write down the expression of E_i in terms of k_i:

$$E_i = k_i - 1. \tag{22}$$

Substituting equation (22) into equation (20) gives $C_i = 2k_i^{-1}$, indicating that the local clustering scales as $C(k) \sim k^{-1}$. Then,

$$C = \frac{2}{N} \sum_{i=1}^{N} \frac{1}{k_i}, \tag{23}$$

we further rewrite equation (23) in a continuous form as

$$C = 2 \int_{k_{\min}}^{k_{\max}} \frac{P(k)}{k} \, dk, \tag{24}$$

where $k_{\min}$ and $k_{\max}$ are the minimal and maximal degrees respectively, $P(k) = \theta k^{-\gamma}$ with $\gamma = 2 + m(1 + \alpha m)^{-1}$ and θ a constant, which is determined by the normalization equation $\int_{k_{\min}}^{k_{\max}} P(k) \, dk = 1$. It is clear that $k_{\min} = 2$ and $k_{\min} \ll k_{\max}$ for sufficiently long time evolution. Then we can deduce the approximate analytical result $C(\alpha, m)$ such that

$$C(\alpha, m) = \frac{\gamma - 1}{\gamma} = \frac{1 + m(\alpha + 1)}{2 + m(\alpha + 2))}. \tag{25}$$

This value is quite large and depends on α and m. Figure 3 shows C as a function of α for different m. However, there exist some errors between the analytical and numerical results. One key reason is that the degree distribution $P(k)$ does not follow a power law for small k, and the nodes with small degree have a high local clustering and a large percentage in the whole network. In other words, though equation (19) can characterize the key property of degree distribution, there is difficulty in computing the clustering coefficient directly using the degree distribution obtained above.

Table 1. Clustering coefficients of general random PNs for various m, where $\alpha = 0$ and network size $N = 10^5$.

m	$P(2)$		$P(3)$		$\ldots$	$\alpha = 0$	
	TR[a]	SR[b]	TR	SR	$\ldots$	TR	SR
1	0.5000	0.5000	0.2000	0.2002	$\ldots$	0.7392	0.7396
2	0.6000	0.6001	0.2000	0.2001	$\ldots$	0.8118	0.8122
3	0.6667	0.6666	0.1905	0.1906	$\ldots$	0.8529	0.8526
4	0.7143	0.7144	0.1786	0.1785	$\ldots$	0.8794	0.8798
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

[a] Theoretical results.
[b] Simulation results.

Consider a special case $\alpha = 0$. Here the rate equation [5] is adopted to solve the computation of clustering coefficient. In general random PNs, we denote the average number of nodes with degree k by $n(k, t)$ at time t, then,

$$n(k + 1, t + 1) = \frac{k}{S_e(t)} n(k, t) + \left(1 - \frac{k + 1}{S_e(t)}\right) n(k + 1, t), \tag{26}$$

where $S_e(t) = (m + 1)t + 1$ is the edge sum of the network at t time step, $n(k, t) \approx mt P(k)$ for large t. And the first term on the right-hand side of equation (26) accounts for the process in which a node with k degree is connected to the new node, leading to a gain in the number of nodes with $k + 1$ degree. A corresponding role is played by the second loss term on the right-hand side of equation (27). Therefore, we derive the iterative expression

$$P(k + 1) = \frac{k}{k + m + 2} P(k) \tag{27}$$

for $k \geqslant 2$. When $k \gg 2$, $P(k) \sim k^{-\gamma}$ with $\gamma = m + 2$. And $P(2)$ is determined by the following

$$m(t + 1)P(2) = m - \frac{2mt P(2)}{S_e(t)} + mt P(2), \tag{28}$$

where the first term on the right-hand side of equation (28) accounts for the continuous introduction of new nodes. Solving equation (28) gives

$$P(2) = \frac{m + 1}{m + 3}. \tag{29}$$

From the above discussions, we derive the same exponent γ for $\alpha = 0$ via continuous approximation and rate equation. However, for small k, the two analytical results show quite different in computing clustering coefficient. To explain clearly, we here compute the clustering coefficient $C(0, m)$ by using the results in equations (27) and (28),

$$C(0, m) = \sum_{k=k_{\min}}^{k_{\max}} P(k)c(k), \tag{30}$$

where $c(k) = 2/k$ is the local clustering coefficient of a node with k degree. Note that in general random PNs, nodes whose degrees are the same have the same local clustering coefficients. Table 1 shows the analytical and numerical results. From table 1, we can see the analytical results match the simulation very well.

3.3. Average distance

The distance of node pairs in network means the minimum number of edges connecting the pair of nodes. The average distance of the whole network can be considered as average over all distances of pairs of nodes. From the view of mathematical description, a logarithmic (or slower than logarithmic) average distance is the most important property of a small-world network.

In the following, let d_{ij} denote the distance between nodes N_i and N_j, $L(t)$ be the average distance of the whole network at time step t. Then the total distance can be written as $\sigma(t) = \sum_{i<j} d_{ij}(t) = \frac{1}{2}m^2t^2L(t)$ at time step t. Here we give a simple proof since the detail is similar to the [15, 28].

In general random PNs, the distances between node pairs already present in the network will not be affected by addition of new nodes and edges. Thus we have

$$\sigma(t+1) = \sigma(t) + \sum_{j=1}^{m}\sum_{i=1}^{E} d_{i,E+j}, \tag{31}$$

where $E = mt + 2$ is the number of edges at time step t. Assume that nodes N_{E+j} with $j = 1, 2, \ldots, m$ are added to the $(m+2)$-regular polygon whose nodes are labelled as N_{x_i} with $i = 1, 2, \ldots, m$, and node pairs N_{x_1}, N_{E+1} and N_{x_m}, N_{E+m}, are respectively connected by corresponding edges. Following from this assumption, we have

$$d_{i,E+j} = \min\{d_{i,x_1} + d_{x_1,E+j} + j, d_{i,x_m} + d_{x_m,E+j} + m + 1 - j\}, \tag{32}$$

for all $j = 1, 2, \ldots, m$.

Furthermore,

$$d_{i,E+j} < D_{i,x} + m, \tag{33}$$

where $D_{i,x} = \min\{d_{i,x_1}, d_{i,x_m}\}$. Substituting equation (33) into equation (31) gives

$$\sigma(t+1) < \sigma(t) + \sum_{i=1}^{E} D_{i,x} + m^2E. \tag{34}$$

Consider the second term on the right-hand side of equation (34) can be approximated as the total distance from one node x to all the other nodes in general random PNs at $t-1$ time step, namely,

$$\sum_{i=1}^{E} D_{i,x} \approx \frac{2\sigma(t-1)}{m(t-1)} < \frac{2\sigma(t)}{mt}. \tag{35}$$

The total distance and average distance increase monotonously with time step. As a result, the inequality in equation (35) holds.

Substituting equation (35) into equation (34), we have

$$\sigma(t+1) < \sigma(t) + \frac{2\sigma(t)}{mt} + m^3t + 2m^2. \tag{36}$$

If equation (35) is written as an equation, then the continuous approximation is

$$\frac{d\sigma(t)}{dt} = \frac{2\sigma(t)}{mt} + m^3t + 2m^2, \tag{37}$$

which leads to a solution such that

$$\sigma(t) = \begin{cases} t^2 \ln t - 2t + c_1t^2 & m = 1 \\ 8t^2 + 8t \ln t + c_2t & m = 2 \\ 2(m-1)^{-1}m^3t^2 + 2(m-2)^{-1}m^3t + c_3t^{\frac{2}{m}} & m > 2 \end{cases} \tag{38}$$
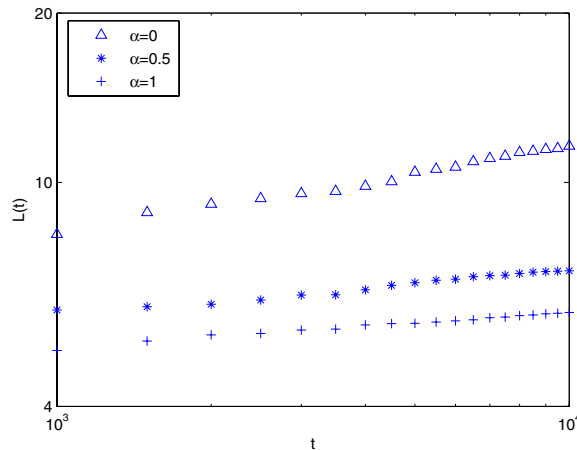
Figure 4. The dependence between average distance $L(t)$ and the network size t of the general random PNs with $m = 2$ and $\alpha = 0$ ($\triangle$), $\alpha = 0.5$ ($*$) and $\alpha = 1$ ($+$). The figure shows that $L(t)$ is almost linear in log–log coordinate.

where c_1, c_2 and c_3 are all constants. When $m = 1$, the average distance increases logarithmically with the network size. For others, the increasing tendency of average distance $L(t)$ is a little slower than $\ln t$. Figure 4 shows the numerical simulations. Obviously, in general random PNs, average distance $L(t)$ grows slower than $\ln t$ as t increases.

4. Conclusion

In summary, we have presented a general random PN model by a simple rule. The rule is realized by assigning fitness to each edge, in which the fitness is related to the arc length of a ring. Continuous growth and connection, governed by the rule, will evolve a general random PN. From the above discussions, we have obtained both analytically and numerically the solutions that the general random PN model exhibits scale-free properties with $\gamma = 2 + m(1 + \alpha m)^{-1}$. By adjusting α and m, we can derive a variety of scale-free networks. In the meantime, the proposed model shows a small-world effect, i.e., the general random PNs have a large clustering coefficient with a hierarchical structure as $c(k) \sim k^{-1}$, and a small average distance scaling slower logarithmically with network size. All these typical properties confirm the real-world networks.

Acknowledgments

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